A Proposed Absolute Entropy of Near Extremal Kerr-Newman Black Hole

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Abstract

Some problems have been found as to the definition of entropy of black hole $S = \frac{A}{4}$ being applied to the extremal Kerr-Newman case, which has conflicts with the third law of thermodynamics. We have proposed a new modification for the near extremal one, which not only obeys the third law, but also does not conflict with other conclusions in black hole thermodynamics. Then we proved that the inner horizon has temperature $T_- = \frac{\kappa_-}{2\pi}$ and proposed that the inner horizon contributes to the entropy of the near extremal one so that the entropy of it is assumed to be $S = (A_+ + A_-)/4$ and vanishes at absolute zero temperature.

Key words: Extremal black hole , entropy , third law of thermodynamics , event horizon

1 Introduction

The thermodynamic properties of black hole have long been researched by scientists. The radiation temperature was regarded as the surface gravity at the outer horizon [1]. For a Schwarzschild black hole, it is

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}, \tag{1}$$

where M is the mass of the black hole. And the entropy of black hole was regarded as the area of the outer event horizon [2] :

$$S = \frac{A}{4}, \tag{2}$$

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$$A = 8\pi\lfloor M^2 + M(\sqrt{M^2 - J^2/M^2 - Q^2}) - Q^2/2\rfloor, \tag{3}$$

where J, Q are the angular momentum and electric charges of black hole. Also, scientists have established the four laws of black hole thermodynamics relative to the four laws in thermodynamics [1,3,4]. And S. Hawking discovered the radiation of black hole; thus the temperature of black hole got a real significance. The temperature of black hole has been widely researched these years but the study of entropy might not be sufficient [5-9]. However,the entropy in formula (2) does not obey the third law of thermodynamics [10,11],which is Nerst's theorem: according to Planck gauge, the entropy of a system will vanishes when its temperature approaches to absolute zero. Actually, according to formula (2) and (3), when $T = (\frac{\partial M}{\partial S})_{J,Q} \to 0$ (the so called "extremal black hole") , we have $S \to \pi\sqrt{Q^4 + 4J^2} \neq 0$. So this definition ($S = \frac{A}{4}$) was unable to be applied to the extremal case , therefore is not the Planck absolute entropy near zero temperature. In order to solve this problem, some scientists, for instance, Hawking, Teitelbom, and Gibbons etc.suggested to redefine the entropy of a near extremal black hole [12-17], and thought it should be zero. Meanwhile, some scientists, such as Loranz, Hiscock, Zaslavskii etc. [18-23] insist on the former definition.Now it is still being discussed. However, we agree with the first group of scientists and redefine the absolute entropy of Kerr-Newman black hole near zero temperature, making it not only obey the third law, but also be harmonious with other conclusions in black hole thermodynamics.

2 The Entropy of Near Extremal Kerr-Newmann Black Hole Derived From Thermodynamics

The entropy we want to find should at least accord with the following three conditions: 1. Obey the Bekenstein-Smarr function:

$$dM = TdS + \Omega dJ + VdQ; \tag{4}$$

2. Obey the limit relations between T and S :

$$S \to 0 (when T \to 0); \tag{5}$$

3. The formula will be reduced to $S = \frac{A}{4}$ when$J, Q \to 0$. Accordingly, by function (4)(condition 1), we have

$$T = \left(\frac{\partial M}{\partial S}\right) s_{J,Q} = \left(\frac{\partial M}{\partial A}\right)_{J,Q} \left(\frac{\partial A}{\partial S}\right)_{J,Q}. \tag{6}$$

And from formula (3), we have

$$\left(\frac{\partial M}{\partial A}\right)_{J,Q} = \frac{1}{4}\left[\frac{1}{8\pi M} - \left(\frac{Q^4}{4} + J^2\right)\frac{8\pi}{MA^2}\right]. \tag{7}$$

In order to accord with condition 3, we suppose the entropy has the form:

$$S = \frac{A - A_0}{4}, \tag{8}$$

where A_0 is a correction item, it may be the function of M ,J ,Q and vanishes when J and Q vanishes(condition 2). From formula (8), we have

$$\left(\frac{\partial A}{\partial S}\right)_{J,Q} = 4 + \left(\frac{\partial A_0}{\partial M}\right)_{J,Q}\left(\frac{\partial M}{\partial S}\right)_{J,Q}. \tag{9}$$

From equation (6) and (9), we have

$$\left(\frac{\partial A}{\partial S}\right)_{J,Q} = 4 + T\left(\frac{\partial A_0}{\partial M}\right)_{J,Q}. \tag{10}$$

Putting equation (7) and (10) into (6), we have

$$\frac{\partial A_0}{\partial M} = \frac{32\pi MA^2}{A^2 - 16\pi^2(Q^4 + 4J^2)} - \frac{4}{T}. \tag{11}$$

From formula (8) and in order to accord with the condition 2 , we get when $T \to 0, A \to A_0$. So A_0 equals to the area of outer horizon at zero temperature. From formula (3), we have

$$A_0 = 4\pi\sqrt{Q^4 + 4J^2}. \tag{12}$$

So that

$$\frac{\partial A_0}{\partial M} = 0. \tag{13}$$

From formulas (3), (8) and (12), we have

$$S = 4\pi M\sqrt{M^2 - J^2/M^2 - Q^2}(T \to 0). \tag{14}$$

From equation (11) and (13), we get

$$T = \frac{A^2 - 16\pi^2(Q^4 + 4J^2)}{8\pi M A^2} = \frac{A^2 - A_0^2}{8\pi M A^2}, \tag{15}$$

which one can prove is exactly the Hawking temperature. When $T \to 0$, we have $A \to A_0$, so that$S \to 0$, it obeys condition 2.And for an uncharged non-rotating black hole, formula (14) will be reduced to formula (2).Thus we think formula (14) may be the absolute Planck entropy of near extremal Kerr-Newman black hole approaching to zero temperature.

3 The Temperature, Heat Capacity and Radiation Power of Near Extremal Black Hole

Formula (15) can be written in another form:

$$T = \frac{1}{8\pi M}(1 - \frac{A_0^2}{A^2}), \tag{16}$$

witch will be reduced to formula (1) for an uncharged non-rotating black hole. From formula (16) we get the heat capacity

$$C_V = C_A = \left(\frac{\partial M}{\partial T}\right)_A = -\frac{8\pi M^2}{\left(1 - \frac{A_0^2}{A^2}\right)}, \tag{17}$$

which will also be reduced to

$$C_V = -8\pi M^2 \tag{18}$$

for an uncharged non-rotating black hole . Moreover, from formula (16) we get the radiation power

$$P = \frac{\partial M}{\partial t} = \sigma A T^4 = \frac{\sigma A}{(8\pi)^4 M^4}\left(1 - \frac{A_0^2}{A^2}\right)^4, \tag{19}$$

where σ is the Stefan-Boltzman constant. For an uncharged non-rotating black hole, we get

$$P = \frac{\sigma}{(8\pi)^3 M^2}. \tag{20}$$

We can see from this section that the redefinition of the entropy for the bear extremal Kerr-Newman black hole in section 2 does not conflict to other thermodynamic properties of non-extremal, uncharged or non-rotating ones .

4 The Physical Interpretation of the Entropy of Near Extremal Kerr-Newman Black Hole

Now we would like to study the microcosmic mechanism of the entropy production of near extremal Kerr-Newman black hole . If we choose the coordinates $x^\nu = (t, r, \theta, \phi)$, we have the Kerr-Newman metric describing the geometry of a rotating charged black hole as follows:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr - Q^2}{\Sigma^2}\right) dt^2 + \frac{\Sigma^2}{\Delta} dr^2 + \Sigma^2 d\theta^2 \\ &+ \left[(r^2 + a^2)\sin^2\theta + \frac{(2Mr - Q^2)a^2\sin^4\theta}{\Sigma^2}\right] d\phi^2 \\ &- \frac{2(2Mr - Q^2)a\sin^2\theta}{\Sigma^2} dt d\phi, \end{aligned} \tag{21}$$

in which we use the standard notations

$$\Sigma^2 = r^2 + a^2\cos^2\theta, \Delta = r^2 - 2Mr + a^2 + Q^2, a = J/M. \tag{22}$$

The surface gravity at the outer ($r = r_+ = M + \sqrt{M^2 - a^2 - Q^2}$) or inner ($r = r_- = M - \sqrt{M^2 - a^2 - Q^2}$) horizon is the limit of the intrinsic acceleration b multiplied by the redshift factor$-\sqrt{g_{00}}$:

$$\kappa_\pm = \lim_{r \to r_\pm} (-b\sqrt{g_{00}}) = \frac{r_+ - r_-}{2(r_\pm^2 + a^2)}. \tag{23}$$

The charged particle in curved space-time obeys Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}} \left(\frac{\partial}{\partial x^\mu} - ieA_\mu\right) \left[\sqrt{-g} g^{\mu\nu} \left(\frac{\partial}{\partial x^\nu} - ieA_\nu\right) \Phi\right] = \mu_0^2 \Phi, \tag{24}$$

whereμ_0 and e are the static masss and electric charge of the particle, and A_μ is the vector potential of the electromagnetic field [9]:

$$A_\mu = -\frac{Qr}{\Sigma}(1, 0, 0, -a\sin^2\theta). \tag{25}$$

Using the metrics (21) and formula (25), the Klein-Gordon equation (24) is reduced to

$$\left[\frac{\left(r^2+a^2\right)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2}\Phi-2a\left[1-\frac{\left(r^2+a^2\right)}{\Delta}\right]\frac{\partial}{\partial t}\frac{\partial}{\partial\phi}\Phi$$
$$-\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r}\Phi-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Phi+\left(\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right)\frac{\partial^2}{\partial\phi^2}\Phi$$
$$+2ie\frac{Qr}{\Delta}\left[a\frac{\partial}{\partial\phi}+(r^2+a^2)\frac{\partial}{\partial t}\right]\Phi+\left(\mu_0^2\Sigma^2-\frac{e^2Q^2r^2}{\Delta}\right)\Phi=0. \tag{26}$$

By decomposing Φ into $\Phi=e^{-i\omega t}u(r)S(\theta)e^{m\varphi}$ (here m is the angular momentum of the particle parallel to the black hole rotating axis and ω is the energy of the particle) and separating the variables, equation (26) is then reduced to the equation governing the angular function

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}S(\theta)+\left[\lambda-\left(\omega a\sin\theta-\frac{m}{\sin\theta}\right)^2-\mu a^2\cos^2\theta\right]S(\theta)=0 \tag{27}$$

and the one governing the radial function

$$\Delta\frac{d^2}{dr^2}u(r)+2(r-M)\frac{d}{dr}u(r)=(\lambda+\mu_0^2r^2-\frac{K^2}{\Delta})u(r)=0, \tag{28}$$

where

$$K=(r^2+a^2)\omega-am-eQr \tag{29}$$

and λ is a constant. If we use the tortoise coordinates transformation:

$$r_\star=\pm\left[r+\frac{1}{2\kappa_+}\ln\frac{|r-r_+|}{r_+}-\frac{1}{2\kappa_-}\ln\frac{|r-r_-|}{r_-}\right], \tag{30}$$

where '+' is for the region $r>r_+$ and ' - ' is for the region$r<r_-$, function (28) will be reduced to

$$\left(r^2+a^2\right)^2\frac{d^2}{dr_\star^2}u(r_\star)+2r\Delta\frac{d}{dr_\star}u(r_\star)=\left[\Delta(\lambda+\mu_0^2r^2)-K^2\right]u(r_\star)=0. \tag{31}$$

When$r\to r_\pm$, we have$\Delta\to 0$, so function (31) is reduced to

$$\frac{d^2}{dr_\star^2}u(r_\star)+\frac{K^2}{\left(r_\pm^2+a^2\right)^2}u(r_\star)=0, \tag{32}$$

which equals to

$$\frac{d^2}{dr_\star^2}u(r_\star) + \left(\omega^2 - \omega_\pm\right)^2 u(r_\star) = 0, \tag{33}$$

where $\omega_\pm = m\Omega_\pm + eV_\pm, \Omega_\pm = \frac{a}{r_\pm^2 + a^2}, V_\pm = \frac{Qr_\pm}{r_\pm^2 + a^2}$.

$\Omega_\pm$ are the angular velocities at the outer and inner horizon and $V_\pm$ are the static electric potentials on the two polar points ($\theta = 0 and \pi$) at the outer and inner horizon respectively. When $r \to r_-$, the solution to the radial function (33) is

$$u = e^{\pm i(\omega - \omega_-)r_\star}, \tag{34}$$

so we have solved the outgoing wave

$$u^{out} = e^{-i\omega t + i(\omega - \omega_-)r_\star} = e^{-i\omega\nu}(r < r_-) \tag{35}$$

and the ingoing wave

$$u^{in} = e^{-i\omega t - i(\omega - \omega_-)r_\star} = e^{-i\omega\nu} \cdot e^{-2i(\omega - \omega_-)r_\star}(r < r_-), \tag{36}$$

where we use the retarded Eddington coordinate:

$$\nu = t - \frac{\omega - \omega^-}{\omega} r_\star. \tag{37}$$

When $r \to r_-$, we see from the tortoise coordinates transformation (30) that

$$r_\star \to \frac{1}{2\kappa_-}\ln(r_- - r), \tag{38}$$

so solution (36) is reduced to

$$u^{in} = e^{-i\omega\nu} \cdot (r_- - r)^{-i(\omega - \omega_-)/\kappa_-}. \tag{39}$$

We then extend the solution (39) to the region $r \geq r_-$ by analytic continuation:

$$\begin{aligned} u^{in} &= L(r_- - r)e^{-i\omega\nu} \cdot (r_- - r)^{-i(\omega - \omega_-)/\kappa_-} \\ &+ L(r - r_-)e^{-i\omega\nu + \pi(\omega - \omega_-)/\kappa_- \cdot (r - r_-)^{-i(\omega - \omega_-)/\kappa_-}}, \end{aligned} \tag{40}$$

where

$$L(r) = \{ \begin{matrix} 1 & r \geq 0 \\ 0 & r < 0 \end{matrix} \quad , \tag{41}$$

so the overall ingoing wave could be written as

$$\begin{aligned} u_\omega = N_\omega \{ L(r_- - r) e^{-i\omega\nu} \cdot (r_- - r)^{-i(\omega - \omega_-)/\kappa_-} \\ + L(r - r_-) e^{-i\omega\nu + \pi(\omega-\omega_-)/\kappa_-} \cdot (r - r_-)^{-i(\omega-\omega_-)/\kappa_-} \}, \end{aligned} \tag{42}$$

where $N_\omega$is a normalization factor, and N_ω^2 stands for the spectrum of radiation. Because $(u_\omega, u_\omega) = N_\omega^2 (1 \pm e^{2\pi(\omega-\omega_-)/\kappa_-}) = \pm 1$, where '+ ' is for fermions and ' - ' is for bosons , so we obtain

$$N_\omega^2 = \frac{1}{e^{(\omega-\omega_-)/T_-} \pm 1}, \tag{43}$$

$$T_- = \frac{\kappa_-}{2\pi}. \tag{44}$$

Thus we proved that the inner horizon of the Kerr-Newman black hole has the temperature $T_- = \frac{\kappa_-}{2\pi}$, and there are particles generated by vacuum polarization erupted from the inside (the region $r < r_-$) to the inner horizon. This makes a new kind of radiation (or absorption[24]). The particle that arrives at the inner horizon will travel to the outer horizon and reduce its temperature to $T_+ = \frac{\kappa_+}{2\pi}$. The particle then goes ahead to erupt from the outer horizon, which makes the usual Hawking radiation. The Bekenstein-Smarr equation of the black hole could be written simply as formula (4). Thus we get $S = \frac{A}{4}$, where A is the area of outer horizon. Also, the Bekenstein-Smarr equation could be written in anothor form:

$$dM = \frac{\kappa_-}{8\pi} dA_- + \Omega_- dJ + V_- dQ, \tag{45}$$

where A_- is the area of inner horizon :

$$A_- = -4\pi(r_-^2 + a^2). \tag{46}$$

We can assume that the contribution to the entropy from the outer and inner horizon are S_+ and S_- respectively:

$$S_\pm = \frac{A_\pm}{4}, \tag{47}$$

so the total entropy is

$$S = S_+ + S_- = 4\pi M\sqrt{M^2 - a^2 - Q^2}, \tag{48}$$

which is the same with formula (14), and obviously, vanishes as temperature vanishes. we therefore think the entropy of the extremal Kerr-Newman black hole is zero and the third law of thermodynamics is still applicable in this case.

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